

# A COMPLETE LIST OF CONSERVATION LAWS FOR NON-INTEGRABLE COMPACTON EQUATIONS OF $K(m, m)$ TYPE

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**ABSTRACT.** In 1993, P. Rosenau and J. M. Hyman introduced and studied Korteweg-de-Vries-like equations with nonlinear dispersion admitting compacton solutions,  $u_t + D_x^3(u^n) + D_x(u^m) = 0$ ,  $m, n > 1$ , which are known as the  $K(m, n)$  equations. In the present paper we consider a slightly generalized version of the  $K(m, n)$  equations for  $m = n$ , namely,  $u_t = aD_x^3(u^m) + bD_x(u^m)$ , where  $m, a, b$  are arbitrary real numbers. We describe all generalized symmetries and conservation laws thereof for  $m \neq -2, -1/2, 0, 1$ ; for these four exceptional values of  $m$  the equation in question is either completely integrable ( $m = -2, -1/2$ ) or linear ( $m = 0, 1$ ). It turns out that for  $m \neq -2, -1/2, 0, 1$  there are only three symmetries corresponding to  $x$ - and  $t$ -translations and scaling of  $t$  and  $u$ , and four nontrivial conservation laws, one of which expresses the conservation of energy, and the other three are associated with the Casimir functionals of the Hamiltonian operator  $\mathfrak{D} = aD_x^3 + bD_x$  admitted by our equation. Our result provides *inter alia* a rigorous proof of the fact that the  $K(2, 2)$  equation has just four conservation laws found by P. Rosenau and J. M. Hyman.

## INTRODUCTION

The equations possessing soliton solutions with compact support (the so-called *compactons*) are presently of great interest for both mathematicians and physicists, see e.g. [9, 10, 12, 13, 14], because such equations can provide adequate models for natural phenomena with a finite span. Initially compactons emerged as solutions of fully nonlinear Korteweg-de-Vries-like equations (the  $K(m, n)$  equations):

$$u_t + D_x^3(u^n) + D_x(u^m) = 0,$$

which have first appeared in [14]; here  $D_x$  denotes the total  $x$ -derivative

$$D_x = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} u_{j+1} \frac{\partial}{\partial u_j},$$

$m, n > 1$ ,  $t$  is the time and  $x$  is the space variable,  $u_j$  denotes  $j$ th derivative of  $u$  with respect to  $x$ ,  $u_0 \equiv u$ , see e.g. [6, 8, 5] for further details on this notation.

Although the solitary waves have compact support only if  $n > 1$  and a compacton is a solution for a  $K(m, n)$  equation in the classical sense only for  $n \leq 3$  [14], it is natural to study a slightly more general version of these equations which we hereinafter refer to as *generalized  $K(m, n)$  equations*:

$$u_t = aD_x^3(u^n) + bD_x(u^m), \tag{1}$$

where  $a, b, m, n$  are arbitrary real numbers.

If  $m = n$ , these equations are easily seen to be Hamiltonian with respect to the Hamiltonian operator  $\mathfrak{D} = aD_x^3 + bD_x$ , the Hamiltonian functional being  $\mathcal{H} = \int \int u^m du dx$ . Thus, equation (1) for  $m = n$  can be written as

$$u_t = aD_x^3(u^m) + bD_x(u^m) = \mathfrak{D}\delta\mathcal{H}, \tag{2}$$

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where  $\delta$  denotes the variational derivative of a functional with respect to  $u$  and  $\int dx$  is understood as a formal integral in the sense of calculus of variations, see e.g. [3, 8] for details. Recall (see e.g. [8]) that for any functional  $\mathcal{F} = \int f(x, t, u, \dots, u_s) dx$  with a smooth density  $f$  we have

$$\delta \mathcal{F} = \frac{\delta f}{\delta u}, \quad \text{where} \quad \frac{\delta}{\delta u} = \sum_{j=0}^{\infty} (-D_x)^j \frac{\partial}{\partial u_j}.$$

The pseudo-differential operator  $\mathfrak{D}^{-1}$  is easily seen to be a formal conservation law of rank  $\infty$  for (2) in the sense of [8]. This means [6, 8] that an infinite set of “standard” obstacles for existence of infinitely many conservation laws of increasing order for (2) vanishes, and therefore one could expect that this equation should share at least some of the properties of integrable PDEs. However, from the results of [6] it can be inferred that only the equations  $K(-2, -2)$  and  $K(-\frac{1}{2}, -\frac{1}{2})$  are integrable (cf. [12] for an alternative argument and Proposition 1 below).

Thus, there are no other nonlinear symmetry integrable equations among those of the generalized  $K(m, m)$  type, i.e., of the form (2). Furthermore, we were able to obtain a complete description of generalized symmetries, including those explicitly dependent on  $t$  and  $x$ , for (2) with  $m \neq -2, -1/2, 0, 1$ , see Proposition 2 below and subsequent discussion.

Knowing these in conjunction with the Hamiltonian operator  $\mathfrak{D}$  has allowed us to provide in Theorem 2 below a complete description of the conservation laws for (2) with  $m \neq -2, -1/2, 0, 1$ . It turns out that, apart from the energy, all conserved functionals for our generalized  $K(m, m)$  equation are Casimir functionals for the Hamiltonian operator  $\mathfrak{D}$ , and therefore they are [8] conserved functionals for any evolution equation of the form  $u_t = \mathfrak{D}f$ , where  $f$  is an arbitrary smooth function of  $x, u$  and a finite number of  $u_j$ .

Note that in [14] four conservation laws for the  $K(2, 2)$  equation were found and it was claimed that no other exist. To the best of our knowledge this claim hasn’t yet been proved. Our results provide *inter alia* a rigorous proof of this assertion.

Recall that the conservation laws could be employed e.g. for the study of stability and for the proof of existence and uniqueness of the solution which belongs to a given function space. Knowing a complete list of conservation laws for a given equation is required, for instance, in order to find normal forms thereof with respect to low-order conservation laws [11], and for constructing higher-precision discretizations, because it is desirable that the latter preserve all conservation laws of the equation in question, cf. e.g. [4] and references therein.

## 1. PRELIMINARIES

Recall (cf. e.g. [2, 6, 8]) that a (smooth) function  $f$  is called *local* if it depends only on  $x, t, u$  and a finite number of  $u_j$ .

Consider an evolution equation with the local right-hand side of the form

$$u_t = F(x, u, u_1, \dots, u_k), \quad k \geq 2. \quad (3)$$

A local function  $G = G(x, t, u, u_1, \dots, u_s)$  is called (see e.g. Chapter 5 of [8] for details) a *characteristic of generalized symmetry* for (3) if it satisfies the linearized version of the latter, that is,

$$D_t(G) = D_F(G),$$

where  $D_t = \partial/\partial t + \sum_{j=0}^{\infty} D_x^j(F) \partial/\partial u_j$  is the total  $t$ -derivative, and  $D_F = \sum_{j=0}^k \partial F/\partial u_j D_x^j$ .

Next, a *formal symmetry of order  $q$*  for (3) is [6, 5] a formal series of the form

$$\mathfrak{L} = \sum_{j=-\infty}^1 a_j D_x^j,$$

where  $a_j$  are local functions, such that

$$\deg(D_t(\mathfrak{L}) - [D_F, \mathfrak{L}]) \leq k + 1 - q.$$

Here the symbol  $\deg$  stands for the degree of a formal series; recall that for  $\mathfrak{M} = \sum_{j=-\infty}^m b_j D_x^j$  with  $b_m \neq 0$  we have  $\deg \mathfrak{M} = m$  by definition, cf. e.g. [6, 5].

Finally, (3) is called *symmetry integrable* [6, 8, 5] if it admits an infinite sequence of explicitly time-independent generalized symmetries of increasing order.

To prove the claims made in Introduction, we shall first need the following result based on the symmetry approach to integrability, see e.g. [5, 6, 7, 16] for details:

**Theorem 1** ([6, 7]). *An equation (3) possesses an explicitly time-independent formal symmetry of order  $N > k$  if and only if the first  $N - k$  canonical densities  $\rho_i$ ,  $i = -1, 0, 1, 2, \dots, N - k - 2$ , are densities of local conservation laws.*

*Existence of an explicitly time-independent formal symmetry of order  $q > N$  is a necessary condition for (3) to possess explicitly time-independent generalized symmetries with the characteristic of order  $q$ .*

Hence, existence of an explicitly time-independent formal symmetry of infinite order is a necessary condition for (3) to be symmetry integrable.

As for the canonical densities for (3), these can be computed recursively from a formal symmetry  $\mathfrak{L}$  of sufficiently high order; explicit formulas for a few of them can be found in [5, 6, 7, 16]. For instance, for  $k > 2$  we have

$$\rho_{-1} = (\partial F/\partial u_k)^{-1/k} \text{ and } \rho_0 = (\partial F/\partial u_{k-1})/(\partial F/\partial u_k). \quad (4)$$

## 2. MAIN RESULTS

**Proposition 1.** *If  $m \neq -2, -1/2, 0, 1$ , then the corresponding generalized  $K(m, m)$  equation (2) has no explicitly time-independent generalized symmetries of order greater than 3; in particular, equation (2) is not symmetry integrable.*

*Proof.* Applying (4) to (2) we obtain that  $\rho_{-1} = (amu^{m-1})^{-1/3}$ . Computing the quantity  $\delta D_t(\rho_{-1})/\delta u$  reveals that it is identically equal to zero only for these values of  $m$ , i.e.,  $m = -2, -1/2, 0, 1$ . Hence (cf. e.g. [8, 5]) our  $\rho_{-1}$  is a density for a conservation law of our equation only for  $m = -2, -1/2, 0, 1$ .

By Theorem 1, our equation (2) for  $m \neq -2, -1/2, 0, 1$  cannot have an explicitly time-independent formal symmetry of order greater than 3, and therefore it cannot possess any explicitly time-independent generalized symmetries of order greater than 3 and, in particular, it cannot be symmetry integrable.  $\square$

Proposition 1 ensures non-existence of *explicitly time-independent* generalized symmetries of order greater than 3. However, this result can be further generalized to explicitly time-dependent symmetries:

**Proposition 2.** *If  $m \neq -2, -1/2, 0, 1$ , then the corresponding generalized  $K(m, m)$  equation (2) has no generalized symmetries, including explicitly time-dependent ones, of order greater than 3.*

*The only generalized symmetries of (2) for  $m \neq -2, -1/2, 0, 1$  are those with the characteristics  $Q_1 = u_x$ ,  $Q_2 = u_t$  and  $Q_3 = (m - 1)tu_t + u$ , i.e.,  $x$ - and  $t$ -translations and the scaling symmetry.*

*Proof.* Indeed, for  $m \neq -2, -1/2, 0, 1$  the above  $\rho_{-1}$  is not a conserved density for (2). In conjunction with Theorem 2 of [15] this is readily seen to imply that (2) has no formal symmetry, even explicitly time-dependent one, of order greater than 3. It is now readily seen (cf. also Theorem 1 in [15]) that (3) has no generalized symmetries, including explicitly time-dependent ones, of order greater than 3.

Now that we know that all generalized symmetries of (2) with  $m \neq -2, -1/2, 0, 1$  are of order at most 3, we can readily find all of them. It turns out that for  $m \neq -2, -1/2, 0, 1$  equation (2) has three symmetries with the characteristics  $Q_1 = u_x$ ,  $Q_2 = u_t$  and  $Q_3 = (m - 1)tu_t + u$ .  $\square$

Note that there are no conservation laws associated (through the Hamiltonian operator  $\mathfrak{D}$ ) to the first and third symmetry. The conserved functional associated to the second symmetry through  $\mathfrak{D}$  is the energy  $\int \int u^m du dx$ .

**Theorem 2.** *If  $\rho$  is a density of a local conservation law for a generalized  $K(m, m)$  equation, where  $m \neq -2, -1/2, 0, 1$ , then it is, up to the addition of a trivial density, a function of  $x, t$  and  $u$  only.*

*Proof.* Let  $\rho$  be a density of a local conservation law for a non-integrable case of the  $K(m, m)$  equation, i.e., for  $m \neq -2, -1/2, 0, 1$ . Then the function  $\gamma = \frac{\delta \rho}{\delta u}$  is a cosymmetry for (2). As the  $K(m, m)$  equation (2) is Hamiltonian with respect to the Hamiltonian operator  $\mathfrak{D} = aD_x^3 + bD_x$ , see (2), and thanks to the fact that Hamiltonian operators map cosymmetries to symmetries, see e.g. [1], we conclude that  $\mathfrak{D}(\gamma)$  is a symmetry of our equation (2). But it follows from Proposition 2 that our equation cannot have generalized symmetries of order greater than 3, so the order of  $\mathfrak{D}(\gamma)$  is less than or equal to 3. Now suppose that the order of the function  $\gamma$  is equal to  $K > 0$ . Then the order of  $\mathfrak{D}(\gamma)$  would be equal to  $3 + K > 3$ , which is a contradiction, so the order of any cosymmetry  $\gamma$  of (2) with  $m \neq -2, -1/2, 0, 1$  must be zero, i.e., it may depend at most on  $x, t, u$ . To any such cosymmetry  $\gamma = \gamma(x, t, u)$  there corresponds a conserved density  $\rho = \int \gamma du$  for which  $\delta \rho / \delta u = \gamma$  by construction. Quite obviously, this  $\rho$  also depends at most on  $x, t, u$  and is defined up to the addition of an arbitrary (smooth) function of  $x$  and  $t$ .  $\square$

Knowing from Theorem 2 that up to the addition of a trivial density any conserved density for (2) with  $m \neq -2, -1/2, 0, 1$  depends at most on  $x, t, u$ , we can readily find all conservation laws of all  $K(m, m)$  equations for  $m \neq -2, -1/2, 0, 1$ . We omit the straightforward computations and state just the relevant result:

**Theorem 3.** *The only local conservation laws of the form  $D_t(\rho) = D_x(\sigma)$  for the generalized  $K(m, m)$  equation (2) with  $m \neq -2, -1/2, 0, 1$ , are, modulo the addition of trivial conservation laws, just the linear combinations of the four conservation laws which for  $b \neq 0$  are given by the formulas*

$$\begin{aligned} \rho_1 &= \int u^m du & \sigma_1 &= \left( mau_{xx}u^{2m-1} + \frac{am(m-2)}{2}u_x^2u^{2m-2} + \frac{b}{2}u^{2m} \right) \\ \rho_2 &= u & \sigma_2 &= aD_x^2(u^m) + bu^m \\ \rho_3 &= u \sin \left( \frac{\sqrt{b}}{\sqrt{a}}x \right) & \sigma_3 &= aD_x^2(u^m) \sin \left( \frac{\sqrt{b}}{\sqrt{a}}x \right) - \sqrt{ab}D_x(u^m) \cos \left( \frac{\sqrt{b}}{\sqrt{a}}x \right) \\ \rho_4 &= u \cos \left( \frac{\sqrt{b}}{\sqrt{a}}x \right) & \sigma_4 &= aD_x^2(u^m) \cos \left( \frac{\sqrt{b}}{\sqrt{a}}x \right) + \sqrt{ab}D_x(u^m) \sin \left( \frac{\sqrt{b}}{\sqrt{a}}x \right). \end{aligned}$$

If  $b = 0$ , then the conservation law with the density  $\rho_3$  is trivial, and the densities  $\rho_2$  and  $\rho_4$  coalesce. However, there are two other conservation laws in such a case, namely

$$\begin{aligned} \rho_5 &= xu & \sigma_5 &= aD_x^2(xu^m) - 3aD_x(u^m) \\ \rho_6 &= x^2u & \sigma_6 &= aD_x^2(x^2u^m) + 6au^m - aD_x(xu^m), \end{aligned}$$

i.e., for  $b = 0$  equation (2) with  $m \neq -2, -1/2, 0, 1$  also has, up to the addition of trivial conservation laws, just four conservation laws with the densities  $\rho_1, \rho_2, \rho_5, \rho_6$  and the fluxes  $\sigma_1, \sigma_2, \sigma_5, \sigma_6$ .

Note that if  $a$  and  $b$  have different signs then sines and cosines of a complex variable appear in the formulas for  $\rho_3, \rho_4, \sigma_3$  and  $\sigma_4$ . In this case it is convenient to divide  $\rho_3$  by the imaginary unit  $i$  and use the following *real* densities and fluxes instead of the above  $\rho_3, \rho_4, \sigma_3$  and  $\sigma_4$ :

$$\begin{aligned} \tilde{\rho}_3 &= cu \sinh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}}x \right) & \tilde{\sigma}_3 &= caD_x^2(u^m) \sinh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}}x \right) - \sqrt{|ab|}D_x(u^m) \cosh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}}x \right) \\ \tilde{\rho}_4 &= u \cosh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}}x \right) & \tilde{\sigma}_4 &= aD_x^2(u^m) \cosh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}}x \right) - c\sqrt{|ab|}D_x(u^m) \sinh \left( \frac{\sqrt{|b|}}{\sqrt{|a|}}x \right), \end{aligned}$$

where  $c = 1$  if  $a > 0$  and  $b < 0$ , and  $c = -1$  if  $a < 0$  and  $b > 0$ .

The conserved functional corresponding to the first conserved density is the energy, i.e., the integral of motion associated with the invariance under the time shifts. If  $m = 2k - 1$  where  $k \in \mathbb{Z} \setminus \{0, 1\}$ , then fact that the quantity  $\int u^{m+1}dx$  is conserved immediately implies the following property of the solutions of the corresponding  $K(m, m)$  equation: if a solution  $u(x, t)$  of (2) belongs to the space  $L^{2k}(\mathbb{R})$  as a function of  $x$ , i.e.,  $\int_{\mathbb{R}} |u|^{2k}dx < \infty$ , at the time  $t = t_0$  then  $u(x, t) \in L^{2k}(\mathbb{R})$  for all  $t \geq t_0$ .

The remaining conserved functionals are Casimir functionals corresponding to our Hamiltonian operator  $\mathfrak{D}$ , so finding a suitable physical interpretation thereof is rather unlikely.

As a final remark, note that it would be interesting to apply our method for proving nonexistence of higher conservation laws using existence of a Hamiltonian operator to other nonintegrable systems.

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